JOURNAL OF APPROXIMATION THEORY 42, 4–13 (1984)

On Set-Valued *f*-Projections and *f*-Farthest Point Mappings*

D. V. PAI

Department of Mathematics, Group of Theoretical Studies, Indian Institute of Technology, Powai, Bombay 400 076, India

AND

P. GOVINDARAJULU†

Department of Mathematics, Shri Venkateswara University, Tirupathi, A.P., India

Communicated by E. W. Cheney

Received July 13, 1982; revised September 2, 1983

Let V be a nonempty closed subset of a separated locally convex space X. Given a lower semi-continuous quasi-convex function f defined on X, one defines here the so-called f-projection $P_{f,V}$. Likewise, given an upper semi-continuous quasi-convex function f on X, one defines here the so-called f-farthest point mapping $Q_{f,V}$. In this exposition, properties of V related to the f-projection $P_{f,V}$ and the f-farthest point mapping $Q_{f,V}$ are defined and several relationships between these properties and continuity of the mappings $P_{f,V}$, $Q_{f,V}$ are explored.

1. INTRODUCTION

Let X, Y be a pair of linear spaces put in duality by a separating bilinear form $\langle\ ,\ \rangle$ and equipped with locally convex topologies compatible with the pairing. Let f be a lower semi-continuous (abbr. l.s.c.) (resp., upper semi-continuous (abbr. u.s.c.)) quasi-convex function defined on X and satisfying $f(\theta) = 0$. Recall that (cf. Daniel [6, p. 14]) f is said to be quasi-convex if the sub-level sets $S_\lambda := \{x \in X: f(x) \leqslant \lambda\}$ are convex for each $\lambda \in \mathbb{R}$. Given a nonempty closed subset V of X and $x \in X$, let $f_V(x)$ (resp. $f^V(x)$) denote the number: $\inf\{f(x-v): v \in V\}$ (resp. $\sup\{f(x-v): v \in V\}$), possibly $= -\infty$ (resp. ∞). Let $P_{f,V}(x)$ (resp. $Q_{f,V}(x)$) denote the set $\{v \in V: f(x-v) =$

* This was the title of the talk of the first author in The Second Edmonton Conference on Approximation Theory, 1982.

† Research of the first author was partially supported by BRNS Grant No. 26/5/82-G.

0021-9045/84 $3.00

$f_V(x)\}$ (resp. the set $\{v \in V: f(x-v) = f^V(x)\}$), eventually void. The set-valued mapping $P_{f,V}$ (resp. $Q_{f,V}$) is called *f-projection* (resp. *f-farthest point mapping*) supported on V. V is said to be *f-proximinal* (resp. *f-Chebyshev*) if $P_{f,V}(x) \neq \varnothing$ (resp. $P_{f,V}(x)$ is a singleton) for each $x \in X$. Likewise, V is said to have the *f-farthest point property*, abbr. *(f-FP)-property* (resp. *f-unique farthest point property*, abbr. *(f-UFP)-property*) if $Q_{f,V}(x) \neq \varnothing$ (resp. $Q_{f,V}(x)$ is a singleton) for each $x \in X$. In case X is a normed space with the norm topology and f is the given norm, there has been a lot of interest in studying properties of the supporting set related to its f-projection (the so-called metric projection in this case). A fairly up-to-date account of this appears in the excellent survey article of Vlasov [18] (also, cf. Singer [17]). In this case there has also been some recent interest in studying analogous properties of sets related to farthest point mappings (e.g., cf. [1, 2, 7, 11, 12, 14]).

In case f is a sub-linear function, properties of sets related to f-projections and f-farthest point mappings have been investigated in [9] and [10], respectively. The principal aim of the present exposition is to obtain results in the same spirit as in [9] and [10] when f is either a quasi-convex function or a convex function satisfying $f(\theta) = 0$. The key tools required for the purpose are collected in Section 2. These are employed to f-projections in Sections 3 and 4 and to f-farthest point mappings in Section 5.

2. Preliminary Results

Let $f: X \to \mathbb{R}$ be a continuous convex function satisfying $f(\theta) = 0$. For $r \in \mathbb{R}, r > 0$, let $S_r := \{x: f(x) \leqslant r\}$ denote the sub-level subset of f. S_r is a convex absorbing set containing the origin θ in its interior. Let $p_r(x) := \inf\{\lambda > 0: x \in \lambda S_r\}$ $(x \in X)$ denote the Minkowski guage of S_r. Then p_r is a nonnegative continuous sublinear function. Given a nonempty closed subset V of X we continue to employ the same terminology as in Section 1 with f replaced by p_r, such as the terms p_r-proximinal, p_r-Chebyshev, etc.

LEMMA 2.1. *Let* $0 < r_1 < r_2$, *then*

(i) $S_{r_1} \subset S_{r_2} \subset \dfrac{r_2}{r_1} S_{r_1}$;

(ii) $p_{r_1} \geqslant p_{r_2} \geqslant \dfrac{r_1}{r_2} p_{r_1}$.

Moreover, if f is sub-linear, then

(iii) $S_{r_1} = \dfrac{r_1}{r_2} S_{r_2}$ *and* $p_{r_1} = \dfrac{r_2}{r_1} p_{r_2}$.

Proof. This is evident. ∎

LEMMA 2.2. *Suppose the convex function f satisfies the property.*
There exists a continuous bijection $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ :=$ *the set of nonnegative reals) such that*

$$f(\lambda x) = \psi(\lambda) f(x) \qquad (\lambda \geqslant 0 \quad and \quad x \in X). \tag{$*$}$$

Then one has

$$S_r = 1/\lambda\, S_{\psi(\lambda) r}$$

and

$$p_r = \lambda p_{\psi(\lambda) r} \qquad (r > 0, \lambda > 0).$$

Furthermore, for $r > 0$, $p_r(x) = 0$ *implies* $f(x) = 0$.

Proof. In view of $(*)$, for $r > 0$ and $\lambda > 0$, the equality $S_r = 1/\lambda\, S_{\psi(\lambda) r}$ is obvious. Also,

$$\begin{aligned} p_r(x) &= \inf\{\alpha > 0 : x \in \alpha S_r\} \\ &= \inf\{\alpha > 0 : x \in \frac{\alpha}{\lambda} S_{\psi(\lambda) r}\} \\ &= \lambda p_{\psi(\lambda) r}(x). \end{aligned}$$

If $p_r(x) = 0$ for some $r > 0$, then there is a sequence $\alpha_n > 0$ such that $x \in \alpha_n S_r$ and $\alpha_n \to 0$. Since $f(x/\alpha_n) = \psi(1/\alpha_n) f(x) \leqslant r$ and $\psi(1/\alpha_n) \to \infty$, we have $f(x) = 0$.

LEMMA 2.3. *Let f be a continuous convex function satisfying* $f(\theta) = 0$ *and let* $r > 0$, *then*

(i) $f(x) \leqslant r \Leftrightarrow p_r(x) \leqslant 1$
(ii) $f(x) = r \Leftrightarrow p_r(x) = 1$
(iii) $f(x) \geqslant r \Leftrightarrow p_r(x) \geqslant 1$.

Proof. This is well known. ∎

PROPOSITION 2.4. *Let f be a continuous convex function satisfying* $f(\theta) = 0$ *and let V be a nonempty closed subset of X. For* $x \in X$,

(i) *if* $P_{f,V}(x) \neq \varnothing$ *and* $f_V(x) = r > 0$, *then* $P_{f,V}(x) = P_{p_r,V}(x)$;
(ii) *if* $Q_{f,V}(x) \neq \varnothing$ *and* $f^V(x) = s > 0$, *then* $Q_{f,V}(x) = Q_{p_s,V}(x)$.

Proof.

$$\begin{aligned} f_V(x) = r > 0 &\Rightarrow f(x-v) \geqslant r, \qquad v \in V \\ &\Leftrightarrow p_r(x-v) \geqslant 1, \qquad v \in V \\ &\Leftrightarrow p_{rV}(x) \geqslant 1 \\ &\Rightarrow p_{rV}(x) = 1, \qquad \text{since } P_{f,V}(x) \neq \varnothing. \end{aligned}$$

Also,

$$\begin{aligned} v_0 \in P_{f,V}(x) &\Leftrightarrow f(x-v_0) = r \\ &\Leftrightarrow p_r(x-v_0) = 1 \\ &\Leftrightarrow v_0 \in P_{p_r,V}(x) \qquad (\text{since } p_{rV}(x) = 1). \end{aligned}$$

This proves (i). The proof of (ii) is identical. ∎

PROPOSITION 2.5. *Let f be a nonnegative continuous convex function satisfying $f(x) = 0$ if and only if $x = \theta$ and property $(*)$. We have*

(i) *If V is f-proximinal, then $P_{f,V} = P_{p_r,V}$ for each $r > 0$. Consequently, V is p_r-proximinal and, moreover, V is p_r-Chebyshev for $r > 0$ if and only if V is f-Chebyshev.*

(ii) *If V satisfies property $(f-FP)$, then $Q_{f,V} = Q_{p_r,V}$ for each $r > 0$. Consequently V satisfies property $(p_r - FP)$ and, moreover, V satisfies property $(p_r - UFP)$ for $r > 0$ if and only if V satisfies property $(f - UFP)$.*

Proof. (i) Let V be f-proximinal. If $f_V(x) = 0$, then $x \in V$ and $P_{f,V}(x) = P_{p_r,V}(x) = \{x\}$ for each $r > 0$. If $f_V(x) = r > 0$, then by Proposition 2.4 (i) $P_{f,V}(x) = P_{p_r,V}(x)$. Let $s > 0$ be given. By property $(*)$ choose $\lambda > 0$ such that $\psi(\lambda) = s/r$ and $f(\lambda x) = \psi(\lambda) f(x)$ for x in X. By Lemma 2.2, $p_r = \lambda p_s$. Therefore $P_{p_r,V} = P_{p_s,V}$ and this entails $P_{f,V} = P_{p_r,V}$ for each $r > 0$. The remaining conclusions are obvious.

(ii) The proof is analogous to that of (i). ∎

By way of an example, $f(x) = x_1^2 + x_1 x_2 + 2x_2^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, satisfies hypothesis of the preceding proposition.

3. ON f-PROJECTIONS

Throughout this section, unless otherwise stated, we assume f to be a l.s.c. quasi-convex function on X (equivalently, the sub-level sets S_λ of f are assumed closed and convex for each $\lambda \in \mathbb{R}$), and V to be a closed subset of X. Recall (cf. Castaing and Valadier [5, Chap. 1]) that f is said to be *inf-*

compact (resp. *inf-locally compact*) if the sub-level sets S_λ are compact (resp. locally compact) for each $\lambda \in \mathbb{R}$, and f is said to be *inf-bounded* if the sub-level sets S_λ are bounded for each $\lambda \in \mathbb{R}$. It is well known (cf. [5, Chap. 1]) that in case f is a l.s.c. convex function, then with the topology $\sigma(X, Y)$ of X, f is inf-compact iff its polar f^* is $\tau(Y, X)$ continuous at θ (here $\tau(Y, X)$ denotes the Mackey topology of Y) and that with any compatible topology of X, f is inf-locally compact iff epi(f) is locally compact (here epi(f) denotes the epigraph of f). Also, in this case f is inf-bounded if $\theta \in$ core dom (f^*) (cf. Rockafellar [15, Theorem 10]).

The set V is said to be *inf-compact* if for each $x \in X$, each minimizing net v_α in V (i.e., a net satisfying $f(x - v_\alpha) \to f_V(x)$) has a convergent subnet in V. f is said to be *strongly quasi-convex* (cf. Daniel [6, p. 15]) if for $x_1, x_2 \in X, x_1 \neq x_2$ and $0 < \lambda < 1, f(\lambda x_1 + (1 - \lambda)x_2) < \max\{f(x_1), f(x_2)\}$.

We remark that with verbatim reproductions of proofs Propositions 2.1 and 2.2 and Theorem 2.2 of [9] extend to the case when f is a l.s.c. quasi-convex function and Corollary 2.2 to the case when f is a strongly quasi-convex function.

In case f is a continuous sub-linear function, it is easily verified that $x \to f_V(x)$ is continuous. Moreover, if we assume V to be inf-compact, then $P_{f,V}$ is u.s.c. (cf. [9, Proposition 2.4]). In what follows, we require the following hypothesis on f (arbitrary) and V:

(H_1) The function $x \to f_V(x)$ is continuous.

(H_2) For $\varepsilon > 0$ and $\alpha > 0$ assigned arbitrarily, there exists $\beta > 0$ such that $f(x) \leqslant \alpha$ and $f(y) \leqslant \beta$ imply $f(x + y) \leqslant \alpha + \varepsilon$.

By way of example, $f(x) = x^2$ (or $f(x) = \sqrt{|x|}$), $x \in \mathbb{R}$ and $V = [a, b]$ satisfy (H_1) and (H_2).

PROPOSITION 3.1. *Let f be a continuous function satisfying hypotheses* (H_1) *and* (H_2) *and let V be inf-compact. Then $P_{f,V}$ is u.s.c.*

Proof. Consider the set $A = \{x \in X: P_{f,V}(x) \cap C \neq \varnothing\}$ for a closed subset C of V. It suffices to prove that A is closed. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net in A such that $x_\lambda \to x_0$. Choose $v_\lambda \in P_{f,V}(x_\lambda) \cap C$, for each $\lambda \in \Lambda$. Then $f(x_\lambda - v_\lambda) = f_V(x_\lambda)$. Hence, by ($H_1$), $\lim_\lambda f(x_\lambda - v_\lambda) = f_V(x_0)$. Let $\varepsilon > 0$ and $\varepsilon_1 > 0$ be arbitrary. Set $\alpha = f_V(x_0) + \varepsilon_1$ and select $\beta > 0$ as given by hypothesis (H_2). There is $\lambda_1 \in \Lambda$ such that $f(x_0 - x_\lambda) \leqslant \beta$ and $f(x_\lambda - v_\lambda) \leqslant \alpha$ for $\lambda \geqslant \lambda_1$. Then by hypothesis ($H_2$),

$$f_V(x_0) \leqslant f(x_0 - v_\lambda) \leqslant \alpha + \varepsilon = f_V(x_0) + \varepsilon_1 + \varepsilon.$$

Therefore, $f_V(x_0) \leqslant \underline{\lim}_\lambda f(x_0 - v_\lambda) \leqslant \overline{\lim}_\lambda f(x_0 - v_\lambda) \leqslant f_V(x_0)$. Whence, $\lim_\lambda f(x_0 - v_\lambda) = f_V(x_0)$ and v_λ is a minimizing net. V being inf-compact,

$\{v_\lambda\}$ has a convergent subnet converging to v_0. C being closed, $v_0 \in C$ and continuity of f entails $v_0 \in P_V(x_0)$. Thus $x_0 \in A$ and A is closed. ∎

THEOREM 3.2. *Let f be a continuous, inf-bounded, quasi-convex function and let V be a closed convex set. Assume that hypothesis (H_1) is fulfilled. Then $P_{f,V}$ is u.s.c. if any of the following conditions hold:*

(1) *f is inf-locally compact;*

(2) *V is locally compact.*

Proof. Suppose that (2) holds. Then by an extension of Proposition 2.2 of [9], V is f-proximinal. Consider the set $A = \{x \in X: P_{f,V}(x) \cap C \neq \emptyset\}$, for C a closed subset of V. Let $\{x_\lambda: \lambda \in \Lambda\}$ be a net in A such that $x_\lambda \to x_0$. It would suffice to prove that $x_0 \in A$. Choose $v_\lambda \in C$ satisfying $f(x_\lambda - v_\lambda) = f_V(x_\lambda)$, for each $\lambda \in \Lambda$. By (H_1), $\lim_\lambda f(x_\lambda - v_\lambda) = f_V(x_0)$. Since $f(-x_\lambda) \to f(-x_0)$, $-x_\lambda$ eventually lies in S_{r_1}, where $r_1 > f(-x_0)$. Likewise, $x_\lambda - v_\lambda$ eventually lies in S_{r_2} for $r_2 > f_V(x_0)$. Since f is quasi-convex, $-v_\lambda = x_\lambda - v_\lambda - x_\lambda$ eventually lies in $2S_r$, for $r = \max\{r_1, r_2\}$. Thus $v_\lambda \in -2S_r \cap V$, eventually. The latter set being closed, bounded, convex and locally compact, it is compact. Thus v_λ has a convergent subnet converging to v. Evidently $v \in P_{f,V}(x_0) \cap C$ and $x_0 \in A$. ∎

Remark. If we add hypothesis (H_2) in the last theorem, then evidently, it is a corollary of Proposition 3.1.

4. ON f-SOLARITY OF SETS

Throughout this section, f will be assumed to be a continuous convex function satisfying $f(\theta) = 0$ and V will be assumed a closed subset of X. Given $\bar{v} \in V$, let Str $(V; \bar{v}) := \bigcup_{v \in V} \{\bar{v} + \lambda(v - \bar{v}): 0 \leqslant \lambda \leqslant 1\}$ denote the star-hull of V at $\bar{v}$. The set V is said to be an *f-sun* (resp. *a strict f-sun*) if for each $x \in X$, $\bar{v} \in P_{f,V}(x_\alpha)$ holds for some (resp. each) element $\bar{v} \in P_{f,V}(x)$ and each $\alpha \geqslant 1$, where $x_\alpha = \bar{v} + \alpha(x - \bar{v})$. In case f is sub-linear or f satisfies the conditions of Proposition 2.5, it is easily verified that V is an f-sun (resp. a strict f-sun) if and only if, for some (resp. each) element $\bar{v} \in P_{f,V}(x)$, we have $\bar{v} \in P_{f,\mathrm{Str}(V;\bar{v})}$. For f sub-linear, this has been observed in [4].

PROPOSITION 4.1. *Let f be as in Proposition 2.5 and let V be f-proximinal. If V is an f-sun (resp. a strict f-sun), then V is a p_r-sun (resp. a strict p_r-sun) for each $r > 0$. Conversely, if V is a p_r-sun (resp. a strict p_r-sun) for some $r > 0$, then V is a f-sun (resp. a strict f-sun).*

Proof. By Proposition 2.5, $P_{f,V} = P_{p_r,V}$ for each $r > 0$ and the proof follows immediately from the definitions. ∎

The proof of the next theorem appears in [8].

THEOREM 4.2. *Let p be a continuous, inf-compact sub-linear function and let V be p-Chebyshev. Then V is a p-sun.*

THEOREM 4.3. *Let f be a nonnegative, continuous convex function which satisfies*

(i) *$f(x) = 0$ if and only if $x = \theta$;*

(ii) *the property $(*)$ of Lemma 2.2;*

(iii) *f is inf-compact.*

Then each f-Chebyshev set is an f-sun.

Proof. Let V be a closed f-Chebyshev set. In view of (i), it suffices to consider $x \in X$ such that $f_V(x) = r > 0$. Let p denote the Minkowski guage of S_r. By Proposition 2.5 $P_{f,V} = P_{p,V}$ and V is p-Chebyshev. Also, by (iii) p is inf-compact, and hence, V is p-sun by the last theorem. By Proposition 4.1, V is an f-sun. ∎

THEOREM 4.4. *Let f be a nonnegative, continuous convex function satisfying* (i), (ii), (iii) *of the last theorem and* (iv) *f is strictly convex and Gâteaux differentiable at each nonzero point of X.*

Then, in X, the class of closed convex sets coincides with the class of closed f-Chebyshev sets.

Proof. If f satisfies (iii) of the last theorem and (iv), then by an extension of Corollary 2.2 of [9], a closed convex set V is f-Chebyshev. Conversely, let V be a closed f-Chebyshev set in X. Then by the last theorem, V is an f-sun. V being f-Chebyshev and f-sun, it follows in view of (i) and (iv) and Proposition 1.3 of [13] that V is convex. ∎

Remark. If there is a continuous, inf-compact convex function f on X satisfying $f(\theta) = 0$, then X is finite dimensional. The above theorem is, therefore, a generalization of the following well-known result of Motzkin, Buseman (cf. Singer [16]): In a smooth and rotund Banach space of finite dimension, the class of Chebyshev sets coincides with the class of closed convex sets.

5. On f-Farthest Point Mappings

As in the last section, we assume for the most part f to be a continuous convex function satisfying $f(\theta) = 0$ and V to be closed subset of X. V is said to be *sup-compact* if each maximizing net $\{v_\alpha\}$ in V (i.e., a net satisfying $f(x - v_\alpha) \to f^V(x)$) has a convergent subnet.

Proposition 5.1. *Let f be an u.s.c. function and V be a sup-compact subset of X. Then V has property $(f - FP)$.*

Proof. This follows immediately from the definitions. ▮

Proposition 5.2. *Let V be a closed set such that $f^V(x) < \infty$ for each $x \in X$ and let f be an u.s.c. function. Consider the following statements:*

(1) *f is inf-compact;*

(2) *V is f-bddly compact;*

(3) *V is sup-compact;*

(4) *V satisfies property $(f - FP)$.*

We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proposition 5.3. *Let f be as in Proposition* 2.5. *Let $v_0 \in Q_{f,V}(x_0)$, for $x_0 \in X$. Then $v_0 \in Q_{f,V}(v_0 + \lambda(x_0 - v_0))$ for all $\lambda \geqslant 1$.*

Proof. For f sub-linear, this is evident. If f satisfies the conditions of Proposition 2.5, then $Q_{f,V} = Q_{p_r,V}$ for each $r > 0$ and the result follows from the sub-linearity of p_r. ▮

In case f is a continuous sub-linear function, it is easily verified that $x \to f^V(x)$ is continuous. Moreover, if V is sup-compact, then $Q_{f,V}$ is u.s.c.

Theorem 5.4. *Let f be a continuous, inf-bounded, quasi-convex function. Let V be a closed convex set such that $f^V(x) < \infty$, for each $x \in X$ and assume the mapping $x \to f^V(x)$ to be continuous. Then $Q_{f,V}$ is u.s.c. if any of the following conditions hold:*

(1) *f is inf-locally compact;*

(2) *V is locally compact.*

Proof. Under the given hypothesis V is inf-compact and by Proposition 5.2, V satisfies property $(f - FP)$. The proof of the proposition is now exactly analogous to the proof of Theorem 3.2. Hence, it is omitted. ▮

Remark. In the preceding theorem, the hypothesis that f is inf-bounded and $f^V(x) < \infty$ for each $x \in X$, may be replaced by: V is bounded and satisfies property $(f-FP)$.

V is said to satisfy *property* (SF) if $x_0 \in X$ and $v_0 \in Q_{f,V}(x_0)$ imply $v_0 \in Q_{f,V}(x_\lambda)$, for each $\lambda, 0 < \lambda < 1$, where $x_\lambda = v_0 + \lambda(x_0 - v_0)$. This property is motivated by Proposition 5.3 and yields an answer to the question: *when is a set satisfying property* $(f-UFP)$ *a singleton*?

THEOREM 5.5. *Let f be as in Proposition* 2.5 *and let V satisfy property* $(f-UFP)$. *Assume that f is inf-compact. Then V satisfies property* (SF).

Proof. By Proposition 2.5, $Q_{f,V} = Q_{p_r,V}$ for each $r > 0$. By Theorem 4.1 of [10], p_r being sub-linear, given $x_0 \in X$ and $v_0 \in Q_{p_r,V}(x_0)$, v_0 is in the set $Q_{p_r,V}(x)$ for each λ, $0 < \lambda < 1$. Hence the same is true for $Q_{f,V}$ and V satisfies property (SF). ∎

PROPOSITION 5.6. *Let f be a nonnegative, continuous convex function satisfying $f(x) = 0$ if and only if $x = \theta$. Then V satisfies property (SF) if and only if V is a singleton.*

Proof. This is evident. ∎

COROLLARY 5.7. *Let f be as in Proposition* 2.5 *and assume that f is inf-compact. Then V satisfies property $(f-UFP)$ if and only if V is a singleton.*

Proof. This follows immediately from Theorem 5.5 and Proposition 5.6. ∎

The preceding corollary extends the main result of [1, Theorem 2].

REFERENCES

1. E. ASPLUND, Sets with unique farthest points, *Israel J. Math.* **5** (1967), 201–209.
2. J. BLATTER, Weiteste punkte und nachste punkte, *Rev. Roumaine Math. Pures Appl.* **14** (1969), 615–621.
3. F. F. BONSALL, "Some Fixed Point Theorems of Functional Analysis," Lecture Notes, Tata Institute of Fundamental Research, Bombay, 1962.
4. W. W. BRECHNER AND B. BROSOWSKI, Ein kriterium zur charakterisierung von sonnen, *Mathematica (Cluj)* **13** (1971), 181–188.
5. C. CASTAING AND M. VALADIER, "Convex Analysis and Measurable Multifunctions," Lecture Notes in Math. No. 580, Springer-Verlag, Berlin/Heidelberg/New York, 1975.
6. J. W. DANIEL, "The Approximate Minimization of Functions," Prentice–Hall, Englewood Cliffs, N.J., 1971.
7. M. EDELSTEIN, Farthest points of sets in uniformly convex Banach spaces, *Israel J. Math.* **4** (1966), 171–176.

8. D. V. PAI AND P. GOVINDARAJULU, On continuity of set-valued f-projections, to appear.
9. P. GOVINDARAJULU AND D. V. PAI, On properties of sets related to f-projections, *J. Math. Ann. Appl.* **73** (1980), 457–465.
10. P. GOVINDARAJULU AND D. V. PAI, On f-farthest points of sets, *Indian J. Pure Appl. Math.* **14** (7) (1983), 873–882.
11. V. KLEE, Convexity of Chebyshev sets, *Math. Ann.* **142** (1961), 292–304.
12. T. D. NARANG, A study of farthest points, *Nieuw Arch. Wisk.* **25** (1977), 54–79.
13. D. V. PAI, Multi-optimum d'une fonctionnelle convexe sur des ensembles réguliers, *C.R. Acad. Sci., Paris, Sér. A*, **280** (1975), 1185–1188.
14. B. B. PANDA AND O. P. KAPOOR, On farthest points of sets, *J. Math. Ann. Appl.* **62** (1978), 345–353.
15. R. T. ROCKAFELLAR, "Conjugate Duality and Optimization," C.B.M.S., Regional Conference Series in Applied Mathematics No. 13, SIAM, Philadelphia, 1974.
16. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag, Berlin/Heidelberg/New York, 1970.
17. I. SINGER, "The Theory of Best Approximation and Functional Analysis," CBMS, Regional Conference Series in Applied Mathematics No. 13, SIAM, Philadelphia, 1974.
18. L. P. VLASOV, Approximative properties of sets in normed linear spaces, *Russian Math. Surveys* **28** (1973), 1–66.